The quantity

$$I(k(\mathbf{x}), G_1, G_2) = \int\limits_{G_2 \setminus G_1} k(\mathbf{x}) |\nabla U_0(\mathbf{x})|^2 d\mathbf{x}$$

is investigated, where $U_0(\mathbf{x})$ is a solution of (2.12). The following assertion holds: among all the surfaces S_1, S_2 bounding the domains G_1, G_2 of a given volume, and among all the equally measurable functions $k(\mathbf{x})$, the minimum $I(k(\mathbf{x}), G_1, G_2)$ is reached in the case when G_1, G_2 are concentric spheres, and the function $k(x)$ is defined in a spherical layer $G_2 \setminus G_1$ which is spherically symmetric and does not decrease as the radius increases.

The boundary value problem (2.12) is encountered, say, in problems of a steady-state temperature of diffusion distribution for non-uniform heat conduction or permeability, respectively, of the medium. The quantity $I(k(\mathbf{x}), G_1, G_2)$ characterizes the heat or mass flow through the surface S_2.

Mathematically, Theorem 4 generalizes the isoperimetric inequality for the electrostatic capacitance /1/ corresponding to the case $k(\mathbf{x}) = \text{const}$.

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SOLID PHASE SEEDS IN A DEFORMABLE MATERIAL*

L.B. KUBLANOV and A.B. FREIDIN

An equilibrium solid phase see in a linearly elastic medium is considered. The problem of a medium with new phase equilibrium domains is reduced to equations of elasticity theory for an inhomogeneous medium with a special kind of definite "phase" deformation under an additional phase equilibrium condition /1/ that imposes a constraint on the shape of the phase boundary.

An ellipsoidal inclusion of an anisotropic phase is considered in an unbounded isotropic medium in a homogeneous external field of stress. It is proved that the tensor being defined by the phase deformation, by a change in the elastic moduli and stresses within the inclusion and having the meaning of a density tensor for dislocation moments indiced by a new phase domain, is global in the case of an equilibrium inclusion. The stress fields in an equilibrium two-phase configuration (TC) are determined by this characteristic property; the surface of the equilibrium ellipsoid turns out to be a surface of equal and constant principal values of the jump of the stress tensor and the constant principal value of the jump of the strain tensor. The stress perturbation tensor deviators within the

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seed and the tensor governing the ellipsoid shape are proportional, which is a generalization of a result obtained for seeds of a melt /2/.

An equation governing the shape and orientation of the ellipsoidal seed as a function of the external stresses and the phase transition parameters follows from the structure of the density tensor of the dislocation moments. The possibility of the existence of an equilibrium ellipsoidal solid phase seed was shown in /3/ where analogous equations were obtained for the case of isotropic phases as a result of solving a TC problem by the method described in /4/; a system of equations is presented in /5/ for the analysis of TC with an anisotropic ellipsoidal seed.

The conditions for the existence of equilibrium seeds and limit configurations analogous to the melt seed configurations are determined /2/.

Energetic changes are considered for TC formation. It is shown that the Gibbs energies of the initial single-phase configuration and the equilibrium TC with an ellipsodial seed are equal; the equilibrium seed turns out to be critical. Seeds can originate only in the metastable phase: for stresses allowing the existence of an equilibrium ellipsoidal seed, the Gibbs specific energy of the initial single-phase configuration is not less than the specific Gibbs energy of a homogeneous configuration in the new phase state, where the equality of these energies is possible only for a TC containing layers. For stresses equal to the stresses within the equilibrium seed the new phase material has a larger specific Gibbs energy than the initial material.

1. Two-phase configuration (TC) in the small strain approximation. We consider an unbounded medium in a homogeneous state of stress σ_0 at a temperature θ and being linearly elastic up to the time of the phase transition. We assume that a phase transition accompanying the "natural" deformation and a change in the elastic moduli occurs in a bounded domain V^+. Additional displacement fields occur here and a stress field is formed that depends on the phase transition characteristics and the shape of the domain V^+.

The problem of an equilibrium TC is to determine the shape of the domain V^+ and the state of stress satisfying the equilibrium conditions /1/ that have the following form in the case of small strains:

$$\nabla\cdot\boldsymbol{\sigma} = 0 \tag{1.1}$$

$$[\mathbf{u}] = 0, \quad \mathbf{n}\cdot[\boldsymbol{\sigma}] = 0, \quad [\theta] = 0 \tag{1.2}$$

$$\rho_0\,[f] - \boldsymbol{\sigma}^{\pm}\cdot\cdot\,[\boldsymbol{\varepsilon}] = 0 \quad (\boldsymbol{\varepsilon} = \operatorname{def}\mathbf{u}) \tag{1.3}$$

and the condition at infinity

$$\boldsymbol{\sigma} \to \boldsymbol{\sigma}_0 \quad \text{as} \quad |\mathbf{x}| \to \infty$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{u}$ is the displacement vector, $\mathbf{n}$ is the unit vector normal to the phase boundary, f is the specific free-energy density, the superscripts "minus" and "plus" denote the material of the initial and new phases, ρ_0 is the material density in the initial state (the "minus" phase with no stresses at the temperature θ), the square brackets denote the change in the quantity during the passage from the "minus" to the "plus" phase, and $\mathbf{x}$ is a point of the body.

Condition (1.1) is satisfied within the phase, and conditions (1.2) and (1.3) on the phase boundary; we neglect the influence of surface tension. It follows from the first condition in (1.2) that

$$[\nabla\mathbf{u}] = \mathbf{n}\mathbf{h}, \quad \mathbf{h} = \mathbf{n}\cdot[\nabla\mathbf{u}] \tag{1.4}$$

The phase equilibrium condition (1.3) is written taking conditions (1.2) and relationships (1.4) into account.

We assume that under given external conditions $(\boldsymbol{\sigma}_0, \theta)$ two single-phase homogeneously stressed configurations exist with strains $\boldsymbol{\varepsilon}_0^{\pm}$ from the initial state. We call these reference configurations. The possibility of their existence means that the equation of state

$$\rho_0 \partial f(\boldsymbol{\varepsilon}, \theta)/\partial\boldsymbol{\varepsilon} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta) \tag{1.5}$$

allows of the solution

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0^-, \theta) = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0^+, \theta) = \boldsymbol{\sigma}_0$$

The specific Gibbs energies of the reference configurations

$$g_0^{\pm} = f(\boldsymbol{\varepsilon}_0^{\pm}, \theta) - \rho_0^{-1}\boldsymbol{\sigma}_0\cdot\cdot\,\boldsymbol{\varepsilon}_0^{\pm}$$

cannot be equal to one another. For instance, for

$$\psi = g_0^- - g_0^+ > 0 \tag{1.6}$$

the reference configuration of the "minus" phase is metastable. The natural strain $[\varepsilon_0]$ generally depends on the stress and temperature at which the phase transition occurs.

The reference configuration concept in the phase transition problem was introduced differently in /4/ where it was assumed that both phases can exist in hydrostatically stressed states which were taken as references. The pressure and temperature selected ensured equality of the specific Gibbs energies of the phases; the external stresses and their corresponding temperature differed from the references.

Assuming the natural strain $[\varepsilon_0]$ and the additional displacement fields $\mathbf{w}$ occurring in the TC to be small, we take a quadratic approximation of the free energy

$$\rho_0 f(\varepsilon^\pm, \theta) = \rho_0 f(\varepsilon^\pm, \theta) + \sigma_0 \cdot\cdot \mathbf{e}^\pm + {}^1/_2 \mathbf{e}^\pm \cdot\cdot \mathbf{C}^\pm \cdot\cdot \mathbf{e}^\pm \tag{1.7}$$
$$\mathbf{e} = \operatorname{def} \mathbf{w}, \quad \mathbf{C}^\pm = (\partial^2 f / \partial \varepsilon \partial \varepsilon)_{\varepsilon_0^\pm, \theta}$$

According to (1.5), the following stresses act in the TC:

$$\sigma^\pm = \sigma_0 + \mathbf{C}^\pm \cdot\cdot \mathbf{e}^\pm \tag{1.8}$$

Since $\sigma_0 = \mathbf{C}_0 \cdot\cdot \varepsilon_0$, where $\mathbf{C}_0 = \mathbf{C}^-$ is the tensor of the elastic moduli in the initial state, the expression for the free energy and Eq.(1.8) can be written in the form

$$\rho_0 f(\varepsilon, \theta) = \begin{cases} \rho_0 f_0^- + {}^1/_2 \varepsilon \cdot\cdot \mathbf{C}_0 \cdot\cdot \varepsilon, & \mathbf{x} \overline{\in} V^+ \\ \rho_0 f_0^+ + {}^1/_2 (\varepsilon - \varepsilon^f) \cdot\cdot \mathbf{C}^+ \cdot\cdot (\varepsilon - \varepsilon^f), & \mathbf{x} \in V^+ \end{cases} \tag{1.9}$$

$$\sigma(\mathbf{x}) = (\mathbf{C}_0 + \mathbf{C}_1 V(\mathbf{x})) \cdot\cdot (\varepsilon(\mathbf{x}) - \varepsilon^f V(\mathbf{x})) \tag{1.10}$$
$$\varepsilon^f = [\varepsilon_0] - \mathbf{B}_1 \cdot\cdot \sigma_0 = \varepsilon_0^+ - \mathbf{B}^+ \cdot\cdot \sigma_0$$
$$\mathbf{C}_1 = \mathbf{C}^+ - \mathbf{C}_0, \quad \mathbf{B}^+ = (\mathbf{C}^+)^{-1}, \quad \mathbf{B}_0 = \mathbf{C}_0^{-1}, \quad \mathbf{B}_1 = \mathbf{B}^+ - \mathbf{B}_0$$
$$f_0^- = f_0(\theta), \quad f_0^+ = f(\varepsilon_0^+, \theta) - {}^1/_2 \sigma_0 \cdot\cdot \mathbf{B}^+ \cdot\cdot \sigma_0$$

Here f_0^- is the specific free-energy density in the initial state and $V(\mathbf{x})$ is the characteristic function of the domain V^+. The strain ε^f is related to the reference configurations; this is the natural strain for a hypothetical transition from the initial state into a new unstressed phase state in which the material has the same properties as the reference configuration of the "plus" phase corresponding to the stresses σ_0; and f_0^+ is the free energy density in this hypothetical state. We note that the "plus" phase cannot exist in the unstressed state.

In combination with conditions (1.1) and the first two conditions of (1.2), Eq.(1.10) reduces the problem of determining the stress field in the given TC to a problem of an inhomogeneous medium with domains having different moduli compared with the surrounding material; the occurrence of these domains is accompanied by the phase strain ε^f. The phase equilibrium condition (1.3) is a constraint on the shape of the domain V^+.

Taking (1.2), (1.9) and (1.10) into account condition (1.3) can be written in the form

$$\gamma + {}^1/_2 ([\sigma] \cdot\cdot \varepsilon^+ - [\varepsilon] \cdot\cdot \sigma^+ - \sigma^+ \cdot\cdot \varepsilon^f) = 0 \quad (\gamma = \rho_0 [f_0]) \tag{1.11}$$

The jumps in the strains and stresses on the boundary of the inclusion in a linearly elastic medium are associated with the stress field within the inclusion and the phase strains by the relationships /6/

$$[\varepsilon] = \mathbf{K}(\mathbf{n}) \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{m}, \quad [\sigma] = \mathbf{S}(\mathbf{n}) \cdot\cdot \mathbf{m} \tag{1.12}$$
$$(\mathbf{K}(\mathbf{n}) = \{\mathbf{n}(\mathbf{n} \cdot \mathbf{C}_0 \cdot \mathbf{n})^{-1} \mathbf{n}\}^s, \quad \mathbf{S}(\mathbf{n}) = \mathbf{C}_0 \cdot\cdot \mathbf{K}(\mathbf{n}) \cdot\cdot \mathbf{C}_0 - \mathbf{C}_0)$$
$$\mathbf{m} = \mathbf{B}_1 \cdot\cdot \sigma^+ + \varepsilon^f = \mathbf{B}_1 \cdot\cdot (\sigma^+ - \sigma_0) + [\varepsilon_0] \tag{1.13}$$

(s denotes symmetrization of the tetravalent tensor during permutation of the subscripts within pairs). It follows from (1.11)-(1.13) that on the equilibrium boundary of the phases

$$\mathbf{m} \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{K}(\mathbf{n}) \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{m} = \varepsilon^+ \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{m} + \sigma^+ \cdot\cdot \varepsilon^f - 2\gamma \tag{1.14}$$

2. Equilibrium ellipsoidal inclusion.

Since the stress field σ^+ within a homogeneous ellipsoidal inclusion in a homogeneous external field is homogeneous /7/, the tensor $\mathbf{m}$ and the right-hand side of (1.14) are constant on the phase boundary. Therefore, the shape of the equilibrium ellipsoidal seed should be such that the stresses within the ellipsoid ensure satisfaction of the following conditions on the boundary:

$$K_* = \mathbf{m} \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{K}(\mathbf{n}) \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{m} = \text{const}\,(\mathbf{n}) \tag{2.1}$$

Assertion. The stresses within an equilibrium ellipsoidal new phase seed in a linearly elastic isotropic medium are such that the tensor $\mathbf{m}$ is spherical:

$$\mathbf{m} = c\mathbf{E} \tag{2.2}$$

($\mathbf{E}$ is the unit tensor and c is a parameter to be determined).

Proof. For an isotropic medium

$$\mathbf{C}_0 = \lambda_0 \mathbf{EE} + 2\mu_0 \mathbf{I}, \quad \mathbf{K}(\mathbf{n}) = (a\mathbf{nEn} - b\mathbf{nnnn})^s$$
$$a = \mu_0^{-1}, \quad b = (\lambda_0 + \mu_0)/[\mu_0 (\lambda_0 + 2\mu_0)] \tag{2.3}$$

where λ_0 and μ_0 are Lamé coefficients and $\mathbf{I}$ is the unit tetravalent tensor /6/. Condition (2.1) can be written in the form

$$K_* = a(\mathbf{n}\cdot\mathbf{q})^2 - b(\mathbf{n}\cdot\mathbf{q}\cdot\mathbf{n})^2 = a\sum_i q_i^2 n_i^2 - b\sum_{i,j} q_i q_j n_i n_j = \mathrm{const}(\mathbf{n}) \tag{2.4}$$

where q_i are the principal values of the tensor $\mathbf{q} = \mathbf{C}_0\cdot\cdot\mathbf{m}$ and n_i are projections of the normal on the principal directions $\mathbf{q}$.

The sufficiency of condition (2.2) for the satisfaction of (2.4) is obvious.

The necessity follows, for instance, from the requirement to satisfy (2.4) at points of the ellipsoid at which the normal is parallel to the principal axes of the tensor $\mathbf{q}$ (where, respectively, $n_i^2 = 1$): $(a - b)\, q_i^2 = \mathrm{const}$, from which $|q_1| = |q_2| = |q_3|$. Let $q_1 = q_2 = -q_3 = q$. Then $K_* = q^2 \{a - b(1 - 2n_3^2)^2\} \neq \mathrm{const}(\mathbf{n})$. Therefore, $q_1 = q_2 = q_3$, i.e., the tensors $\mathbf{q}$ and $\mathbf{m}$ are spherical.

The stress and strain fields within the ellipsoidal inclusion are governed by the equations /6, 7/

$$\boldsymbol{\sigma}^+ = \boldsymbol{\sigma}_0 + \mathbf{C}_0\cdot\cdot(\boldsymbol{\Omega} - \mathbf{I})\cdot\cdot\mathbf{m}, \quad \boldsymbol{\varepsilon}^+ = \boldsymbol{\varepsilon}_0 + \boldsymbol{\Omega}\cdot\cdot\mathbf{m} \tag{2.5}$$

where $\boldsymbol{\Omega}$ is the Eshelby tensor, which depends on the geometrical characteristics of the ellipsoid and the elastic moduli of the surrounding material.

It follows from (1.13) and (2.5) that

$$(\mathbf{C}_1^{-1}\cdot\cdot\mathbf{C}_0 + \boldsymbol{\Omega})\cdot\cdot\mathbf{m} = -\mathbf{B}_0\cdot\cdot\boldsymbol{\sigma}_* \tag{2.6}$$

$$\boldsymbol{\sigma}_* = \mathbf{B}_1^{-1}\cdot\cdot[\boldsymbol{\varepsilon}_0] = \boldsymbol{\sigma}_0 + \mathbf{B}_1^{-1}\cdot\cdot\boldsymbol{\varepsilon}^f \tag{2.7}$$

Substituting (2.2) into (2.6), we obtain an equation to determine the shape of the equilibrium ellipsoidal seed

$$\boldsymbol{\omega} = -c^{-1}\mathbf{B}_0\cdot\cdot\boldsymbol{\sigma}_* + \boldsymbol{\omega}_* \tag{2.8}$$

$$\boldsymbol{\omega}_* = -3k_0\mathbf{C}_1^{-1}\cdot\cdot\mathbf{E} = \mathbf{E} + \mathbf{B}_0\cdot\cdot\mathbf{B}_1^{-1}\cdot\cdot\mathbf{E} \tag{2.9}$$

Here $\boldsymbol{\omega} = \boldsymbol{\Omega}\cdot\cdot\mathbf{E}$ is a tensor coaxial to the ellipsoid; its principal values are /7/

$$\omega_i = \frac{3\varkappa v}{8\pi}\int_0^\infty \frac{du}{(a_i^2 + u)\sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}}$$
$$\varkappa = \mathrm{tr}\,\boldsymbol{\omega} = (1 + \nu_0)/(1 - \nu_0), \quad v = 4\pi a_1 a_2 a_3/3 \tag{2.10}$$

where ν_0 is Poisson's ratio of the "minus" phase, a_i is the ellipsoid semi-axis, and k_0 is the volume compression modulus of the "minus" phase. Since $\mathrm{tr}\,\mathbf{B}_0\cdot\cdot\boldsymbol{\sigma}_* = I_*/(3k_0)$ where $I_* = \mathrm{tr}\,\boldsymbol{\sigma}_*$, it follows from (2.7), (2.9) and (2.10) that

$$c = -I_*/Q, \quad Q = 3k_0(\varkappa - \mathrm{tr}\,\boldsymbol{\omega}_*) \tag{2.11}$$

Condition (2.2) and its corollaries (2.8) and (2.11) are only the necessary conditions for phase equilibrium.

Subsituting (2.2) and (2.3) into (1.14) and (2.5), we obtain an equation connecting the parameter c, the tensors $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}_0$ and the phase transition parameters, from which it follows, by taking account of (2.8) and (2.11), that the equilibrium TC with a new phase ellipsoidal seed can exist in a homogeneous stress field if

$$I_*^2 = -2\gamma_* Q, \quad \gamma_* = \gamma + {}^1/_2\boldsymbol{\varepsilon}^f\cdot\cdot\mathbf{B}_1^{-1}\cdot\cdot\boldsymbol{\varepsilon}^f \tag{2.12}$$

By virtue of (2.12), the phase transition parameters should be such that Q and γ_* have different signs. It follows from (2.11) and (2.12) that

$$c = -\mathrm{sign}\,(QI_*)\sqrt{2\,|\,\gamma_*/Q\,|} \tag{2.13}$$

The second condition for the existence of an ellipsoidal seed ($\mathbf{i}$ is an arbitrary vector)

$$\mathbf{i}\cdot(\boldsymbol{\omega}_* - c^{-1}\mathbf{B}_0\cdot\cdot\boldsymbol{\sigma}_*)\cdot\mathbf{i} \geqslant 0 \tag{2.14}$$

follows from the non-negative definiteness of the tensor $\boldsymbol{\omega}$.

By virtue of relationships (1.12), (1.13), (2.2) and (2.5) the state of stress in a medium with an equilibrium ellipsoidal seed is such that:

1°. The stress and strain perturbation tensors within the seed are coaxial to the ellipsoid. The tensors $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}^* - \boldsymbol{\varepsilon}_0$, and the deviators of the tensors $\boldsymbol{\omega}$ and the stress perturbations (the prime denotes the tensor deviator) are proportional

$$\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}_0 = c\mathbf{C}_0 \cdot\cdot (\boldsymbol{\omega} - \mathbf{E}), \quad \boldsymbol{\varepsilon}^+ - \boldsymbol{\varepsilon}_0 = c\boldsymbol{\omega}$$
$$\boldsymbol{\omega}' = (\boldsymbol{\sigma}^{+\prime} - \boldsymbol{\sigma}_0')/(2\mu_0 c) \qquad (2.15)$$

(relationships analogous to the first two are obtained in /5/, the third generalizes the solution for a melt seed /2/).

2°. The surface of the equilibrium ellipsoid is a surface of equal and constant principal values of the jump in the stress tensor and a constant value of the jump of the strain tensor

$$[\boldsymbol{\sigma}] = 2\mu_0 \varkappa c\,(\mathbf{nn} - \mathbf{E}), \quad [\boldsymbol{\varepsilon}] = c\varkappa\mathbf{nn}$$

3°. The stresses and strains within the seed are determined by the phase transition parameters:

$$\boldsymbol{\sigma}^+ = \mathbf{B}_1^{-1} \cdot\cdot (c\mathbf{E} - \boldsymbol{\varepsilon}^f)$$
$$\boldsymbol{\varepsilon}^+ = -(3k_0 c\mathbf{C}_1^{-1} \cdot\cdot \mathbf{E} + \mathbf{B}_0 \cdot\cdot \mathbf{B}_1^{-1} \cdot\cdot \boldsymbol{\varepsilon}^f) \qquad (2.16)$$

4°. On the ellipsoid surface

$$\sigma^- = c\,\{2\mu_0\varkappa\,(\mathbf{E} - \mathbf{nn}) + \mathbf{B}^{-1} \cdot\cdot \mathbf{E}\} - \mathbf{B}_1^{-1} \cdot\cdot \varepsilon^f$$

(the parameter c is determined by Eq.(2.13)).

It follows from (2.8), (2.13) and (2.15) that the shape and orientation of the ellipsoid are governed by the tensor $\boldsymbol{\sigma}_*$ and the anisotropy of the new phase elastic moduli tensor

$$\omega' = \alpha\xi' + \boldsymbol{\omega}_*', \quad \xi = \boldsymbol{\sigma}_* / I_*, \quad \alpha = Q / (2\mu_0) \qquad (2.17)$$

By virtue of condition (2.14) the domain of existence of the seed in the space of principal values ξ_k of the tensor ξ, the equilateral triangle in the deviator plane

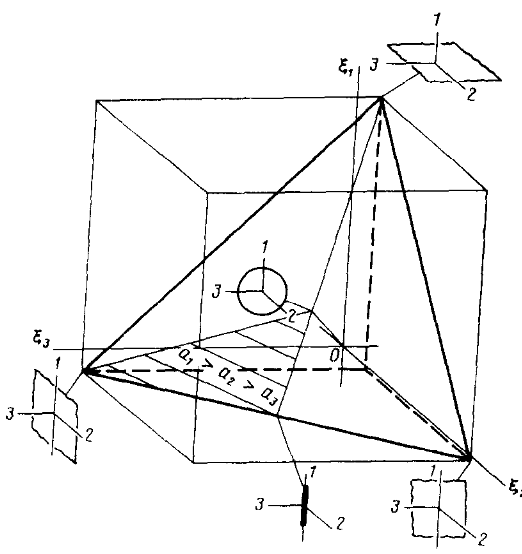
$$\sum_k \xi_k = 1 \qquad (2.18)$$

with vertices on the lines of intersection of the planes $\alpha\xi_k = (\alpha - \varkappa)/3 - \mathbf{i}_k \cdot \boldsymbol{\omega}_*' \cdot \mathbf{i}_k$, where $\mathbf{i}_k$ are the principal directions of the tensor ξ.

If the "plus" phase material is isotropic then $\boldsymbol{\omega}_*' = 0, \boldsymbol{\omega}' = \alpha\xi'$ is an ellipsoid coaxial to the tensor $\boldsymbol{\sigma}_*$. (An analogous equation was obtained in another form in /3/). The ellipsoid is coaxial to the stress tensor at infinity if the tensors $\boldsymbol{\varepsilon}^f$ and $\boldsymbol{\sigma}_0$ are coaxial or $\mu_+ = 0$. It follows from (2.13) and the equality $I_* = \operatorname{tr} \mathbf{B}_1^{-1} \cdot\cdot [\boldsymbol{\varepsilon}_0] = -3\,[\vartheta_0]\,k_+ k_0/k_1$, where, $[\vartheta_0] = \operatorname{tr}\,[\boldsymbol{\varepsilon}_0]$, $k_+ = k_0 + k_1$, that

$$I_* = -\operatorname{sign}\,([\vartheta_0]/k_1)\,\sqrt{2\,|\,\gamma_* Q\,|} \qquad (2.19)$$

Since $Q = 3k_0\,(\varkappa + 3k_0/k_1)$, $\nu_0 < {}^1/_2$, $k_0, k_+ > 0$, then $k_1 Q > 0$ and therefore



$$\operatorname{sign} \alpha = \operatorname{sign} Q = \operatorname{sign} k_1, \quad c = \operatorname{sign} [\vartheta_0] \sqrt{2 \mid \gamma_* / Q \mid} \tag{2.20}$$

Condition (2.14) defines a triangle (figure) obtained when the plane (2.18) intersects the cube

$$(\alpha - \varkappa)/3 \leqslant \alpha \xi_k \leqslant (\alpha + 2\varkappa)/3$$

Ellipsoids of different shape correspond to different points of the triangle.

We assume that $k_1 > 0$ $(\alpha > 0)$. For $\xi_1 = \xi_2 = \xi_3 = {}^1/_3$ a spherical seed is the equilibrium seed. As ξ_1 increases, and for example, corresponding to the decrease $\xi_2 = \xi_3$, the seed transforms into an ellipsoid of revolution flattened in the direction $\mathbf{i}_1$; for the decrease $\xi_1 < {}^1/_3$ the ellipsoid is elongated along the direction $\mathbf{i}_1$. The limit configurations, provided the volume is finite, are planes and infinite spikes. Such limit configurations were described for the melt seed /2/. The case $k_1 < 0$ can be considered analogously.

If the material of the "plus" phase is transversely isotropic, while the tensor $\boldsymbol{\varepsilon}^f$ is axisymmetric ($\boldsymbol{\varepsilon}^f = \varepsilon_1 \mathbf{ll} + \varepsilon_2 (\mathbf{E} - \mathbf{ll})$, where $\mathbf{l}$ is the direction of the isotropy axis, then

$$\begin{aligned}
&\boldsymbol{\omega}' = -\{\boldsymbol{\sigma}_0'/(2\mu_0 c) + d\,({}^1/_3\mathbf{E} - \mathbf{ll})\} \\
&\gamma_* = \gamma + \{(b_{22} + b_{23})\,\varepsilon_1^2 + 2b_{11}\varepsilon_2^2\}\,\delta^{-1} \\
&Q = 3k_0\,(\varkappa - 3) - (b_{22} + b_{23} + 2b_{11})\,\delta^{-1} \\
&d = 2\,\{(b_{22} + b_{23} - b_{12} - b_{11})\,c - (b_{22} + b_{12} + b_{23})\,\varepsilon_1 \\
&\quad (2b_{12} + b_{11})\,\varepsilon_2\}\,\delta^{-1}, \quad \delta = 2b_{11}\,(b_{22} + b_{23}) - 4b_{12}^2 \\
&b_{kk} = \frac{1}{E_k} - \frac{1}{E_0}, \quad b_{12} = \frac{\nu_0}{E_0} - \frac{\nu_1}{E_1}, \quad b_{23} = \frac{\nu_0}{E_0} - \frac{\nu_2}{E_2}
\end{aligned}$$

Here E_0 is Young's modulus of the "minus" phase, E_1 and E_2 are Young's moduli of the "plus" phase under tension along the axis and in the plane of isotropy, respectively, and ν_1 and ν_2 are Poisson's ratios of the "plus" phase characterizing the transverse compression along the isotropy axis and in the isotropy plane under tension in the isotropy plane, respectively. If the isotropy axis is directed along one of the principal directions of the tensor $\boldsymbol{\sigma}_0$, then the ellipsoid is coaxial to $\boldsymbol{\sigma}_0$.

3. Energy changes in TC formation.

Analysis of the stability of equilibrium configurations does not arise in the problem of this paper. Some consequences of comparing the Gibbs energy of single- and two-phase configurations are examined below. The change in Gibbs energy

$$G = \int_V \rho_0 f(\boldsymbol{\varepsilon}, \theta)\, dV - \int_\Gamma \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{u}\, d\Gamma$$

of a body V bounded by a surface Γ when an inclusion V^+ arises isothermally from a new phase material under dead load conditions, can be represented in the form

$$\Delta G = \int_{V^+} (\gamma_* - {}^1/_2 \boldsymbol{\sigma}_* \cdot\cdot\, \mathbf{m})\, dV \tag{3.1}$$

The first two conditions of (1.2), representations of the free energy (1.7) and (1.9) and the equality resulting from (1.10) and (1.13) $\boldsymbol{\sigma}_0 \cdot\cdot\, \boldsymbol{\varepsilon}^+ - \boldsymbol{\sigma}^+ \cdot\cdot\, \boldsymbol{\varepsilon}_0^- = \boldsymbol{\sigma}_0 \cdot\cdot\, \mathbf{m}$ were used in deducing (3.1). The work of the external forces in forming the inclusion under dead loading conditions

$$A = \int_{V^+} \boldsymbol{\sigma}_0 \cdot\cdot\, \mathbf{m}\, dV$$

is determined analogously.

In the case of an ellipsoidal inclusion in a homogeneous field of stress

$$\Delta G = V^+ (\gamma_* - {}^1/_2 \boldsymbol{\sigma}_* \cdot\cdot\, \mathbf{m}) \tag{3.2}$$

$$A = V^+ \boldsymbol{\sigma}_0 \cdot\cdot\, \mathbf{m} \tag{3.3}$$

The equality of the Gibbs energy of the initial stage-phase configuration and the equilibrium TC with an ellipsoidal seed follows from (2.2), (2.11), (2.13), (2.16) and (3.2):

$$\Delta G = 0 \tag{3.4}$$

The work is here determined by the global part of the stress tensor associated with the phase-transition parameters

$$\begin{aligned}
A &= -V^+ Q^{-1} I_* \operatorname{tr} \boldsymbol{\sigma}_0 = V^+ \{2\gamma_* + \operatorname{sign}(I_* Q) \operatorname{tr} \mathbf{B}_1^{-1} \cdot\cdot \\
&\boldsymbol{\varepsilon}^f \sqrt{2 \mid \gamma_* / Q \mid}\,\}
\end{aligned}$$

By virtue of (2.19) and (2.20), for an isotropic seed

$$A = V^{-}\{2\gamma_* + \operatorname{sign}[\vartheta_0]\,(3k_+k_0/k_1)\,\vartheta' \sqrt{2\,|\gamma_*/Q|}\}$$

Taking (2.6) in to account it follows from (3.2) that

$$\partial \Delta G/\partial V^+ = \gamma_* - {}^1/_2\boldsymbol{\sigma}_* \cdot\cdot \mathbf{R} \cdot\cdot \boldsymbol{\sigma}_*, \quad \mathbf{R} = -(\mathbf{C}_1 \cdot\cdot \mathbf{C}_0 + \boldsymbol{\Omega})^{-1} \cdot\cdot \mathbf{B}_0$$

where $\mathbf{R}$ is determined by the shape of the inclusion.

For $\gamma_* > {}^1/_2\boldsymbol{\sigma}_* \cdot\cdot \mathbf{R} \cdot\cdot \boldsymbol{\sigma}_*$ it is energetically preferable to reduce the volume and get rid of the non-equilibrium seed. Such seeds are subcritical. For $\gamma < {}^1/_2\boldsymbol{\sigma}_* \cdot\cdot \mathbf{R} \cdot\cdot \boldsymbol{\sigma}_*$ a non-equilibrium seed expands. The equilibrium configuration (2.17) satisfies condition (3.4) and is critical. The stresses within a solid phase seed depend on its shape outside the relation with the surface tension (see (2.5)); this dependence determines the critical seed configuration just as the pressure difference inside and outside a drop in gas-liquid phase transitions and the critical radius are determined by the surface tension on the phase boundary /8/.

The difference (1.6) of the Gibbs specific energies of the single-phase configurations under stresses $\boldsymbol{\sigma}_0$ satisfying conditions for the existence of an equilibrium ellipsoidal seed can be written in the form

$$\begin{aligned}\psi &= {}^1/_2\boldsymbol{\sigma}_* \cdot\cdot \mathbf{B}_1 \cdot\cdot \boldsymbol{\sigma}_* - \gamma_* = \\ &{}^1/_2c^2\,\{Q + (\boldsymbol{\omega} - \boldsymbol{\omega}_*) \cdot\cdot \mathbf{C}_0 \cdot\cdot \mathbf{B}_1 \cdot\cdot \mathbf{C}_0 \cdot\cdot (\boldsymbol{\omega} - \boldsymbol{\omega}_*)\} = \\ &2\mu_0^2c^2\,\{(\boldsymbol{\omega} - \varkappa\mathbf{E}) \cdot\cdot \mathbf{B}_1 \cdot\cdot (\boldsymbol{\omega} - \varkappa\mathbf{E}) + \varkappa/\mu_0\} = \\ &\mu_0c^2\,\{2\mu_0\,(\boldsymbol{\omega} - \varkappa\mathbf{E}) \cdot\cdot \mathbf{B}^+ \cdot\cdot (\boldsymbol{\omega} - \varkappa\mathbf{E}) + (\varkappa^2 - \boldsymbol{\omega} \cdot\cdot \boldsymbol{\omega})\}\end{aligned}$$

Since $\boldsymbol{\omega} \cdot\cdot \boldsymbol{\omega} \leqslant \varkappa^2$ then

$$\psi \geqslant 2\mu_0^2c^2\,(\boldsymbol{\omega} - \varkappa\mathbf{E}) \cdot\cdot \mathbf{B}^+ \cdot\cdot (\boldsymbol{\omega} - \varkappa\mathbf{E}) > 0$$

since $\mathbf{B}^+$ is a positive-definite tensor. Therefore, the occurrence of equilibrium ellipsoidal seeds is possible only when the initial single-phase configuration is metastable.

The Gibbs energy of a TC with an ellipsoidal seed equals the energies of both single-phase configurations if $\psi = 0$. For $c \neq 0$ this is possible only for a limit configuration containing a layer (a plane for a bounded seed volume) $\omega = \varkappa nn$ under conditions of "plus" phase elastic moduli degeneration: $\mathbf{B}^+ \cdot\cdot (\mathbf{E} - \mathbf{nn}) = 0$. Note that equilibrium TC with new phase layers do not exhaust the limit configurations with equilibrium ellipsoidal seeds.

The least degree of metastability ψ allowing the existence of an equilibrium seed corresponds to stresses for which the quadratic form $(\boldsymbol{\omega} - \varkappa\mathbf{E}) \cdot\cdot \mathbf{B}_1 \cdot\cdot (\boldsymbol{\omega} - \varkappa\mathbf{E})$ is minimal. In the case of isotropic phases this condition has the form

$$-(\mu_1\mu_0/\mu_+)\,\boldsymbol{\omega} \cdot\cdot \boldsymbol{\omega} = \min\;(\mu_1 = \mu_+ - \mu_0)$$

For $\mu_1 < 0$, a spherical seed is equilibrium in the least metastable single-phase configuration $\boldsymbol{\omega} = (\varkappa/3)\,\mathbf{E}$. Here

$$\psi = {}^2/_3\mu_0c^2\varkappa^2\,(1 + 4\mu_0/(3k_+))$$

If $\mu_1 > 0$ then the limit configuration $\boldsymbol{\omega} = \varkappa\mathbf{nn}$ is correspondingly equilibrium. Here

$$\psi = 2\mu_0^2c^2/(\varkappa_+\mu_+) \to 0 \quad \text{as} \quad \mu_+ \to \infty$$

The corresponding stresses are determined by Eq.(2.8).

In conclusion, we note that when using the equality $\mathbf{C}_1^{-1} = -\mathbf{B}^+ \cdot\cdot \mathbf{B}_1^{-1} \cdot\cdot \mathbf{B}_0$ it follows from (2.2), (2.11) and (2.12) that the material within a seed in a homogeneous field of stress $\boldsymbol{\sigma}^+$ has a larger specific Gibbs energy density that the initial material in the same field of stresses

$$g^+(\boldsymbol{\sigma}^+) - g^-(\boldsymbol{\sigma}^+) = \gamma_* - {}^1/_2\mathbf{m} \cdot\cdot \mathbf{B}_1^{-1} \cdot\cdot \mathbf{m} = 3k_0\gamma_*(\varkappa - 3)/Q > 0$$

since $\gamma_*/Q < 0, \varkappa < 3$.

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THE STABILITY OF GROWING INHOMOGENEOUSLY AGEING VISCOELASTIC BODIES*

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The results in /1, 2/ on the stability of growing viscoelastic rods in finite and infinite time intervals are generalized.

1. Formulation of the problem of the stability of a growing viscoelastic body. We consider a body fabricated at a time $t = 0$ and occupying the domain Ω_0 in three-dimensional space. Continuous growth of the body occurs in the time interval $[t_0, t_1]$, where $t_0 \geqslant 0$. The law of growth, i.e., the dependence of the body configuration on time, is considered to be given. The time of generation of a material particle with coordinates $\mathbf{x} = \{x_i\}$ $(i = 1, 2, 3)$ is denoted by $\tau^*(\mathbf{x})$.

The body is subjected to mass loads F and surface loads q applied to the body boundary $S_q(t)$, $\mathbf{F} = \{F_i\}$, $\mathbf{q} = \{q_i\}$. Note that the body surface through which growth of the material occurs is part of the surface S_q. On the other part of the body surface $S_n(t)$ we are given the displacements, which to be specific, we set equal to zero. We will later assume that the type of boundary conditions does not change during body fabrication.

Displacements $u_i(t, \mathbf{x})$ governing the unperturbed trajectory motion appears in the body under the action of external forces. We will henceforth assume the growth of the body to occur fairly slowly and the displacements u_i to be slowly varying functions of time, whereupon inertial effects can be neglected.

We assume that during the growth of the body its configuration turns out to be different from the designed one (for instance, the longitudinal axis of a growing rod actually turns out to be curved instead of straight (designed)). This means that the material point coordinates (when there are no external loads) are $x_i + \alpha v_i^\circ$ instead of x_i. We consider v_i° to be fairly small. The parameter α is introduced provisionally, it can be set equal to unity.

In such a body the displacements will equal $u_i^* = u_i + \alpha v_i$.

We will call the body motion governed by the displacements u_i^* perturbed and the displacements αv_i the desired perturbations.

We introduce the displacement norm ($V(t)$ is the body volume at the time t)

$$\|\mathbf{u}(t)\| = \Big(\int\limits_{V(t)} u_i(t, \mathbf{x})\, u_i(t, \mathbf{x})\, dV\Big)^{1/2}$$

Here and henceforth, summation is over repeated subscripts.

Definition. The unperturbed motion of a growing viscoelastic body is called stable in an infinite time interval if for any number $A > 0$ as small as desired there is a number $\delta = \delta(A) > 0$, such that for any initial displacements αv_i° satisfying the inequality $\alpha \|\mathbf{v}^\circ\| < \delta$ and displacements αv_i corresponding to this perturbation will satisfy the inequality $\alpha \|\mathbf{v}\| < A$ for $0 \leqslant t < \infty$.

If the motion of the growing body is investigated in a finite time interval $[0, T]$ and a critical value is given for the displacement norm $\|\mathbf{v}\|^*$, then it is possible to speak of a

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